

A universal lower bound for the first eigenvalue of the Dirac operator on quaternionic Kähler manifolds

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Abstract

A universal lower bound for the first positive eigenvalue of the Dirac operator on a compact quaternionic Kähler manifold M of positive scalar curvature is calculated. It is shown that it is equal to the first positive eigenvalue on the quaternionic projective space. For this, the horizontal tangent bundle on the canonical $\mathbf{SO}(3)$ -bundle over M is equipped with a hyperkählerian structure and the corresponding splitting of the horizontal spinor bundle is considered. The desired estimate is obtained by looking at hyperkählerian twistor operators on horizontal spinors.

1 Introduction

On a compact Riemannian spin manifold of nonnegative scalar curvature the first positive eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{n-1} \frac{\inf_M \kappa}{4}, \quad (1.1)$$

where κ denotes the scalar curvature of M . This result was found by Friedrich [Fri80], who has also shown that equality is attained if and only if there exist at least one nontrivial Killing spinor on M . Killing spinors are automatically eigenspinors for D of smallest possible eigenvalue.

Afterwards, Hijazi [Hij84] could show that this estimate cannot be sharp if the manifold is Kähler, i.e. the lower bound cannot be attained as eigenvalue of D^2 . An improvement of the estimate in this case was done by Kirchberg [Kir86],

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[Kir90]. He could show that on a compact Kähler manifold M^{2m} of nonnegative scalar curvature one gets as estimates

$$\begin{aligned}\lambda^2 &\geq \frac{m+1}{m} \frac{\inf_M \kappa}{4} \quad \text{for } m \text{ odd}, \\ \lambda^2 &\geq \frac{m}{m-1} \frac{\inf_M \kappa}{4} \quad \text{for } m \text{ even}.\end{aligned}\tag{1.2}$$

In addition, Kirchberg introduced the notion of Kählerian Killing spinors, and he could show that exactly for these spinors equality is attained.

Moreover, Hijazi's theorem [Hij84] says that on a Riemannian spin manifold M^n there cannot be any nontrivial Killing spinor if there is a parallel k -form, $k \neq 0, n$ on M . Hence, Friedrich's estimate is not sharp on quaternionic Kähler manifolds too, because in this case there is a canonical parallel 4-form, namely the Kraines form. Several attempts were made by Hijazi and Milhorat [HiM95-2], [HiM96] to improve the estimate of the first eigenvalue also in this case. As in the case of Kähler manifolds, they tried to introduce a suitable notion of twistor spinors and to use the fact that the spinor bundle $\mathbf{S}(M)$ splits into eigenbundles $\mathbf{S}_r(M)$ under Clifford multiplication with the Kraines form. But up to now all estimates of the first eigenvalue of D^2 depend on the concerning eigenbundle $\mathbf{S}_r(M)$ in which the spinor lives. If M^{4m} is a compact spin quaternionic Kähler manifold of positive scalar curvature, Hijazi and Milhorat could show in [HiM96] that for the eigenvalue λ^2 of an eigenspinor which lives in the bundle $\mathbf{S}_0(M)$ or in the bundle $\mathbf{S}_r(M)$ for $[\frac{m}{2}] + 1 \leq r \leq m$ one has the estimate $\lambda^2 \geq \frac{m+3}{m+2} \frac{\kappa}{4}$ (there is no infimum because on quaternionic Kähler manifolds the scalar curvature is always constant). But in all other eigenbundles the estimate is weaker. In spite of this it was conjectured that the right-hand side of the estimate above gives a universal lower bound, because it is exactly the first eigenvalue of the spectrum of D^2 on the quaternionic projective space [Mil92]. There was the similar situation in the Riemannian case (Friedrich's estimate): lower bound = first eigenvalue on the standard sphere; and in the Kähler case for odd complex dimensions (Kirchberg's estimate): lower bound = first eigenvalue on the complex projective space. As a corollary of the result in [HiM96] it is seen that at least for $m = 2$ and $m = 3$ the mentioned conjecture is correct.

In this paper it will be shown that the conjecture is true for all quaternionic dimensions m :

Theorem 1.1 *Let M^{4m} be a compact spin quaternionic Kähler manifold of positive scalar curvature κ . Then every eigenvalue λ of the Dirac operator satisfies the estimate*

$$\lambda^2 \geq \frac{m+3}{m+2} \frac{\kappa}{4}.\tag{1.3}$$

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2 Quaternionic Kähler manifolds, the bundle P

Quaternionic Kähler manifolds are defined as manifolds having holonomy $\mathbf{Sp}(1) \cdot \mathbf{Sp}(m) = (\mathbf{Sp}(1) \times \mathbf{Sp}(m))/\mathbb{Z}_2$. Therefore there locally exist three almost complex structures J_a , $a = 1, 2, 3$ which satisfy the multiplicative rules of imaginary unit quaternions:

$$J_a J_b = \varepsilon_{abc} J_c - \delta_{ab} . \quad (2.4)$$

In addition, there are corresponding 2-forms (considered as elements in the Clifford algebra)

$$\Omega_a = \frac{1}{2} \sum_{i=1}^{4m} e_i J_a e_i , \quad a = 1, 2, 3 , \quad (2.5)$$

which are also defined only locally. The local almost complex structures span a three dimensional subbundle E of the endomorphism bundle $\text{End}(TM)$ of TM , which is closed under the Levi-Civita connection. All J_a , $a = 1, 2, 3$ and the corresponding 2-forms are not parallel, but on quaternionic Kähler manifolds there is a canonical parallel 4-form, the Kraines form [Bon67], [Kra66]:

$$\Omega = \sum_a \Omega_a \wedge \Omega_a = \sum_a \Omega_a \cdot \Omega_a + 6m . \quad (2.6)$$

Here, for the second equality, the canonical vector bundle isomorphism between $\mathbb{C}\mathbf{I}(M)$ and $\Lambda_{\mathbb{C}}^*(M)$ was used and Proposition 2.3 of [HiM95-2] was applied. A special choice of three local almost complex structures can be identified with a local frame in E . On the space P of all of these frames the group $\mathbf{SO}(3)$ operates in a natural manner; in this way P becomes a principal- $\mathbf{SO}(3)$ -bundle, and let $\pi : P \rightarrow M$ denote the canonical projection. The covariant derivative on E induced by the Levi-Civita connection characterizes a connection form ω on P .

To construct a metric on P , one considers the vector fields on P induced by the action of $\mathbf{SO}(3)$. Let H_a , $a = 1, 2, 3$ be a base of $\mathfrak{so}(3)$, orthogonal with respect to the Killing form, which satisfies the following commutator relation:

$$[H_a, H_b] = 2\varepsilon_{abc} H_c . \quad (2.7)$$

The corresponding vector fields ξ_a , $a = 1, 2, 3$ can be written as

$$\xi_a(p) = \left. \frac{d}{dt} \right|_{t=0} (p \exp(tH_a)) , \quad p \in P . \quad (2.8)$$

Let ω_a be the 1-form dual to ξ_a which annihilates all vectors that are horizontal w.r.t. ω , i.e. ω can be written as

$$\omega = \sum_a c_a \omega_a \quad (2.9)$$

for some $c_a \in \mathbb{R}$. The metric on P can now be defined by

$$g_P := \pi^* g_M + \sum_a \omega_a \otimes \omega_a. \quad (2.10)$$

Hence, there is a orthogonal splitting of the tangent bundle of P into a horizontal and a vertical part:

$$TP = T_H P \oplus T_V P. \quad (2.11)$$

In the future only the horizontal bundle will be of special interest. As abbreviation $HP = T_H P$ will be used. The idea of the proof is to lift all calculations from the quaternionic Kähler manifold to the bundle P as exactly as possible but with the difference that on P there are now globally defined complex structures which in addition are parallel w.r.t. the horizontal connection on HP . That means that the horizontal bundle will be equipped with a hyperkähler structure.

At this point it is convenient to introduce some conventions. If $X \in TM$ is a vector on M , let $X^* \in HP$ denote its horizontal lifting to P . Explicitly one has $X^* \perp T_V P$. The horizontal connection $\tilde{\nabla}$ on HP is defined by

$$\tilde{\nabla}_{X^*} Y^* := (\nabla_X Y)^* \quad \text{and} \quad \tilde{\nabla}_V Y^* = 0 \quad (2.12)$$

for all $V \in T_V P$.

Now one considers the pull-back $\pi^* \mathbf{S}(M)$ of the spinor bundle $\mathbf{S}(M)$ onto P , which is isomorphic to the spinor bundle $\mathbf{S}(HP)$ associated to HP . It is a $\mathbb{C}\mathbf{I}(HP)$ -module in a natural manner, where the Clifford multiplication operates by

$$X^* \pi^* \psi(p) := \pi^* (X\psi)(p), \quad \psi \in \Gamma(\mathbf{S}(M)). \quad (2.13)$$

Moreover, $\mathbf{S}(HP)$ is equipped with the pull-back connection on M :

$$\tilde{\nabla}_{X^*} \pi^* \psi(p) = \pi^* (\nabla_X \psi)(p). \quad (2.14)$$

On sections of $\mathbf{S}(HP)$ a horizontal Dirac operator \tilde{D} is defined by

Definition.

$$\tilde{D}\psi = \sum_{i=1}^{4m} e_i^* \tilde{\nabla}_{e_i^*} \psi, \quad \psi \in \Gamma(\mathbf{S}(HP)). \quad (2.15)$$

Here, $\{e_i^*\}$ is a orthonormal base of HP .

The key point of the following investigations is

Proposition 2.1 *Let $\psi \in \Gamma(\mathbf{S}(M))$ be an eigenspinor of D with eigenvalue λ . Then $\pi^* \psi \in \Gamma(\mathbf{S}(HP))$ is an eigenspinor of \tilde{D} with the same eigenvalue λ .*

Proof. The proof is straightforward because of the definition of the horizontal connection $\tilde{\nabla}$. \square

On HP there are naturally defined global almost complex structures. A point $p \in P$ is a frame of three almost complex structures over a point $m \in M$:

$$p = (J_1(p), J_2(p), J_3(p)). \quad (2.16)$$

Definition. Let J_1, J_2, J_3 be the three almost complex structures on HP , defined by

$$J_a X^*(p) := (J_a(p)X)^*(p). \quad (2.17)$$

Proposition 2.2 J_1, J_2 and J_3 on P are parallel w.r.t. $\tilde{\nabla}$.

Proof. It has to be shown that $\tilde{\nabla}_{X^*} J_1 = 0$ for all $X^* \in HP$. But this is seen at once, because by definition the connection form on P satisfies $\omega(X^*) = 0$. \square

Therefore, three Kähler forms can be defined on P which are denoted in the following by $\tilde{\Omega}_a$, $a = 1, 2, 3$:

$$\tilde{\Omega}_a = \frac{1}{2} \sum_{i=1}^{4m} e_i J_a e_i, \quad (2.18)$$

where $\{e_i^*\}$ is a orthonormal base of HP . In the same manner there exists a horizontal Kraines form on P :

$$\tilde{\Omega} := \sum_a \tilde{\Omega}_a \wedge \tilde{\Omega}_a = \sum_a \tilde{\Omega}_a \cdot \tilde{\Omega}_a + 6m. \quad (2.19)$$

One has $\tilde{\Omega} = \pi^* \Omega$, and $\tilde{\Omega}$ is parallel w.r.t. $\tilde{\nabla}$. Hence HP is equipped with a hyperkähler structure w.r.t. $\tilde{\nabla}$.

3 Splitting of the horizontal spinor bundle

Since HP is the only bundle which is dealt with, the star $*$ that denotes horizontal liftings of vectors will be omitted, and the short notation $\mathbf{S} := \mathbf{S}(HP)$ will be used.

The spinor bundle $\mathbf{S}(M)$ on M splits into eigenbundles of the Kraines form; this was shown by Hijazi and Milhorat [HiM95-1]. This splitting is carried over to \mathbf{S} at once. Besides of the horizontal Krainesform $\tilde{\Omega}$ on P one can choose one of the three horizontal Kählerforms which in the following will always be denoted by $\tilde{\Omega}_1$. $\tilde{\Omega}$ and $\tilde{\Omega}_1$ are parallel w.r.t. $\tilde{\nabla}$. Because of $[\tilde{\Omega}, \tilde{\Omega}_1] = 0$ there is in addition to the mentioned splitting of \mathbf{S} a decomposition into eigenbundles of $\tilde{\Omega}_1$. For further investigation it is necessary to look at representations of $\mathfrak{sl}(2, \mathbb{C})$. The horizontal Kähler forms $\tilde{\Omega}_a$ satisfy the following commutator relations:

$$[\tilde{\Omega}_a, \tilde{\Omega}_b] = 4\varepsilon_{abc} \tilde{\Omega}_c. \quad (3.20)$$

One considers in $\mathbb{C}\mathbf{l}(HP)$ the forms

$$O_a := \frac{i}{2} \tilde{\Omega}_a \quad (3.21)$$

and

$$O_1^+ := \frac{1}{2}(O_2 + iO_3), \quad O_1^- := \frac{1}{2}(O_2 - iO_3). \quad (3.22)$$

It can be verified at once that this is a representation of $\mathfrak{sl}(2, \mathbb{C})$:

$$\begin{aligned} [O_1, O_1^+] &= 2O_1^+ \\ [O_1, O_1^-] &= -2O_1^- \\ [O_1^+, O_1^-] &= O_1. \end{aligned} \quad (3.23)$$

The corresponding Casimir operator is easily written down. W.r.t. the Killing form, $\frac{1}{8}O_1$ is dual to O_1 , and $\frac{1}{4}O_1^+$ resp. $\frac{1}{4}O_1^-$ is dual to O_1^- resp. O_1^+ . Hence the Casimir operator is given by

$$\begin{aligned} C &= \frac{1}{8}O_1O_1 + \frac{1}{4}O_1^+O_1^- + \frac{1}{4}O_1^-O_1^+ = \frac{1}{8} \left(\sum_a O_a O_a \right) \\ &= -\frac{1}{32}(\tilde{\Omega} - 6m). \end{aligned} \quad (3.24)$$

In general, the Casimir operator operates on an irreducible representation of highest weight μ by multiplication with $\|\mu + \rho\|^2 - \|\rho\|^2$, where ρ denotes the half sum of positive roots. The scalar product on the space of weights is defined by $\langle \mu, \nu \rangle := B(t_\mu, t_\nu)$, where t_μ is the uniquely determined element of the Cartan subalgebra \mathfrak{h} with $B(t_\mu, h) = \mu(h)$ for all $h \in \mathfrak{h}$. In the special case $\mathfrak{sl}(2, \mathbb{C})$ (the Cartan subalgebra has dimension 1 and is spanned by O_1) μ is simply a natural number r . Let V be a irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight r , then the vector of highest weight satisfies

$$O_1 v_r = r v_r, \quad (3.25)$$

and C operates on V by multiplication with $\frac{1}{8}r(r+2)$.

Using this the splitting $\mathbf{S}(HP)$ into eigenbundles w.r.t. the operation of $\tilde{\Omega}$ and $\tilde{\Omega}_1$ is determined:

$$\begin{aligned} \mathbf{S} &= \bigoplus_{r=0}^m \mathbf{S}_r, \quad \tilde{\Omega}|_{\mathbf{S}_r} = (6m - 4r(r+2))\text{id}, \\ \mathbf{S}_r &= \bigoplus_{s=0}^r \mathbf{S}_r^{(s)}, \quad \tilde{\Omega}_1|_{\mathbf{S}_r^{(s)}} = i(2r - 4s)\text{id}. \end{aligned} \quad (3.26)$$

To summarize, one gets the following facts: if the partial bundle \mathbf{S}_r with eigenvalue $6m - 4r(r+2)$ w.r.t. $\tilde{\Omega}$ is considered, there is an additional splitting into smaller bundles, which are eigenbundles of the Clifford multiplication with

$\tilde{\Omega}_1$. Between the eigenvalues of $\tilde{\Omega}_1$ on these partial bundles there are gaps of absolute value 4.

On the other side it is known that like spinor bundles on Kähler manifolds the whole of \mathbf{S} splits into eigenbundles of $\tilde{\Omega}_1$ with eigenvalues $i(2m - 2k)$: $\mathbf{S} = \bigoplus_{k=0}^{2m} \mathbf{S}^k$, $k = 0, \dots, 2m$. Explicitly one has

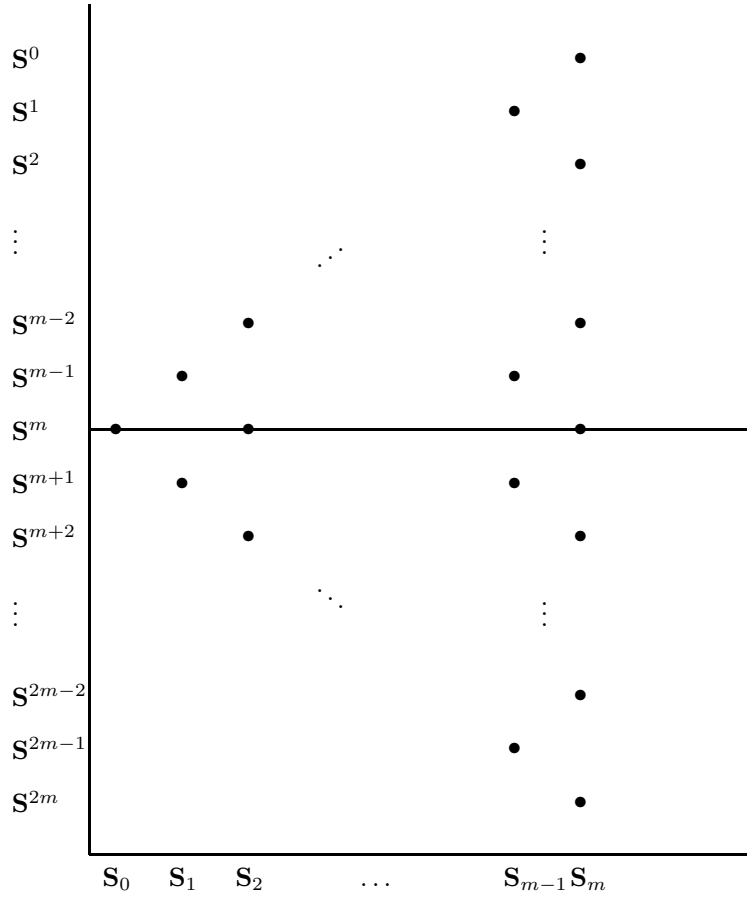
$$\mathbf{S}_r^{(s)} = \mathbf{S}_r \cap \mathbf{S}^{m-r+2s}. \quad (3.27)$$

To avoid inconveniences with notations the definition

$$\mathbf{S}_r^k := \mathbf{S}_r^{(\frac{k+r-m}{2})} = \mathbf{S}_r \cap \mathbf{S}^k \quad (3.28)$$

for $\frac{k+r-m}{2} \in \mathbb{N}_0$ will be used. In the future the appearance of \mathbf{S}_r^k should be interpreted in the sense that all which is said should be ignored if k does not satisfy the integrability condition or if the bundle \mathbf{S}_r^k does not exist at all (if e.g. $k < 0$).

All that was said can be clarified by a picture:



The Clifford multiplication with vectors in HP is not compatible with the splitting. In fact, a spinor lying completely in one distinguished eigenbundle of $\tilde{\Omega}$ resp. $\tilde{\Omega}_1$ will be carried into the direct sum of the two neighbouring eigenbundles:

Lemma 3.1

$$\begin{aligned}\mu : HP \otimes \mathbf{S}_r &\rightarrow \mathbf{S}_{r-1} \oplus \mathbf{S}_{r+1} \quad \text{resp.} \\ \mu : HP \otimes \mathbf{S}^k &\rightarrow \mathbf{S}^{k-1} \oplus \mathbf{S}^{k+1}.\end{aligned}\quad (3.29)$$

Proof. See [Kir90], [HiM96]. \square

Summarizing these facts one gets

$$\mu := \mu^{++} \oplus \mu^{+-} \oplus \mu^{-+} \oplus \mu^{--} : HP \otimes \mathbf{S}_r^k \rightarrow \mathbf{S}_{r+1}^{k+1} \oplus \mathbf{S}_{r+1}^{k-1} \oplus \mathbf{S}_{r-1}^{k+1} \oplus \mathbf{S}_{r-1}^{k-1} \quad (3.30)$$

(here every non-existing summand in the Whitney sum is to be omitted). To project Clifford multiplication on every summand, one has to follow the ideas of Kirchberg and Hijazi and to combine them. The following lemmata are given without proofs. For the proof of Lemma 3.2 see [Kir90] and for the proofs of Lemmata 3.3 and 3.4 see [HiM96]. If the bundle HP is complexified and the Clifford multiplication is linearly extended, the following facts are easily proven:

Lemma 3.2

$$\begin{aligned}q^+(X) &= \frac{1}{2}(X + iJ_1X) \quad : \quad \mathbf{S}^k \rightarrow \mathbf{S}^{k+1} \\ q^-(X) &= \frac{1}{2}(X - iJ_1X) \quad : \quad \mathbf{S}^k \rightarrow \mathbf{S}^{k-1}.\end{aligned}\quad \square \quad (3.31)$$

Hijazi [HiM96] invented the operator \mathcal{J} , which is defined by $\mathcal{J}(X) := \sum_a \tilde{\Omega}_a J_a(X) + 3X$ for the treatment of the splitting w.r.t. $\tilde{\Omega}$:

Lemma 3.3

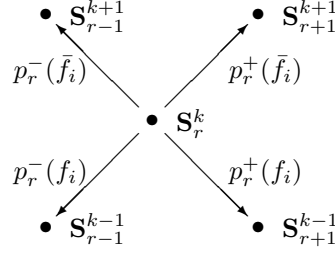
$$\begin{aligned}[\tilde{\Omega}, X] &= 4\mathcal{J}(X) \\ [\tilde{\Omega}, \mathcal{J}(X)] &= -8\mathcal{J}(X) + 12X - 4X(\tilde{\Omega} - 6m).\end{aligned}\quad \square \quad (3.32)$$

With help of this one can define similar projectors in the case of the splitting in eigenbundles w.r.t. the Kraines form:

Lemma 3.4

$$\begin{aligned}p_r^+(X) &= \frac{1}{4(r+1)}((2r+1)X - \mathcal{J}(X)) \quad : \quad \mathbf{S}_r \rightarrow \mathbf{S}_{r+1} \\ p_r^-(X) &= \frac{1}{4(r+1)}((2r+3)X + \mathcal{J}(X)) \quad : \quad \mathbf{S}_r \rightarrow \mathbf{S}_{r-1}.\end{aligned}\quad \square \quad (3.33)$$

Let $\{e_j\}$ be an orthogonal base of HP with the property $e_{2j} = J_1 e_{2j-1}$, $j = 1, \dots, 2m$. $f_j = q^-(e_{2j-1})$ and $\bar{f}_j = q^+(e_{2j-1})$ form the corresponding complex base. The operation of the projectors can be interpreted in a nice graphical way:



4 Hyperkählerian Twistor operators

The twistor operator \mathcal{D} on a Riemannian spin manifold M^n is by definition the composition of the covariant differential ∇ and followed by the orthogonal projection onto the kernel of the Clifford multiplication:

$$\begin{aligned} \mathcal{D}\psi = \text{proj}_{\ker \mu}(\nabla\psi) &= \sum_i e_i \otimes \nabla_{e_i} \psi + \frac{1}{n} \sum_{i,j} e_i \otimes e_i e_j \nabla_{e_j} \psi \\ &= \sum_i e_i \otimes \nabla_{e_i} \psi + \frac{1}{n} \sum_{i,j} e_i \otimes e_i D\psi \end{aligned} \quad (4.34)$$

for an arbitrary orthogonal base $\{e_i\}$. Lower estimates for the first eigenvalue of the Dirac operator are established by considering the inequality $\|\mathcal{D}\psi\|^2 \geq 0$. But if the manifold M carries additional structure, this is not sufficient, as shown by Hijazi [Hij84] in the case of Kähler manifolds. Here it was necessary to split the Clifford multiplication in a similar way as above and to define partial twistor operators.

This approach will be used also in the case of a hyperkähler structure on HP . Corresponding to the splitting (3.30) of Clifford multiplication the horizontal Dirac operator can be splitted into four parts:

$$\tilde{D} = D^{++} + D^{+-} + D^{-+} + D^{--}, \quad (4.35)$$

which, restricted to $\Gamma(\mathbf{S}_r^k)$, have the following form:

$$\begin{aligned} D^{++} &= 2 \sum_{j=1}^{2m} p_r^+(\bar{f}_j) \tilde{\nabla}_{f_j} \\ D^{+-} &= 2 \sum_{j=1}^{2m} p_r^+(f_j) \tilde{\nabla}_{\bar{f}_j} \end{aligned}$$

$$\begin{aligned}
D^{-+} &= 2 \sum_{j=1}^{2m} p_r^-(\bar{f}_j) \tilde{\nabla}_{f_j} \\
D^{--} &= 2 \sum_{j=1}^{2m} p_r^-(f_j) \tilde{\nabla}_{\bar{f}_j}.
\end{aligned} \tag{4.36}$$

Therefore the partial twistor operators are easily written down (again as above, restricted to $\Gamma(\mathbf{S}_r^k)$):

Lemma 4.1

$$\begin{aligned}
\mathcal{D}^{++} &= \sum_{j=1}^{2m} \left(p_r^+(\bar{f}_j) \otimes \tilde{\nabla}_{f_j} \right. \\
&\quad \left. - \frac{1}{A_{r+1,k+1}^{++}} p_r^+(\bar{f}_j) \otimes p_{r+1}^-(f_j) D^{++} \right) \\
\mathcal{D}^{+-} &= \sum_{j=1}^{2m} \left(p_r^+(f_j) \otimes \tilde{\nabla}_{\bar{f}_j} \right. \\
&\quad \left. - \frac{1}{A_{r+1,k-1}^{+-}} p_r^+(f_j) \otimes p_{r+1}^-(\bar{f}_j) D^{+-} \right) \\
\mathcal{D}^{-+} &= \sum_{j=1}^{2m} \left(p_r^-(\bar{f}_j) \otimes \tilde{\nabla}_{f_j} \right. \\
&\quad \left. - \frac{1}{A_{r-1,k+1}^{-+}} p_r^-(\bar{f}_j) \otimes p_{r-1}^+(f_j) D^{-+} \right) \\
\mathcal{D}^{--} &= \sum_{j=1}^{2m} \left(p_r^-(f_j) \otimes \tilde{\nabla}_{\bar{f}_j} \right. \\
&\quad \left. - \frac{1}{A_{r-1,k-1}^{--}} p_r^-(f_j) \otimes p_{r-1}^+(\bar{f}_j) D^{--} \right)
\end{aligned} \tag{4.37}$$

with

$$\begin{aligned}
A_{r,k}^{\pm+} &= \sum_{j=1}^{2m} p_{r\mp 1}^{\pm}(\bar{f}_j) p_r^{\mp}(f_j) \big|_{\mathbf{S}_r^k} \quad \text{and} \\
A_{r,k}^{\pm-} &= \sum_{j=1}^{2m} p_{r\mp 1}^{\pm}(f_j) p_r^{\mp}(\bar{f}_j) \big|_{\mathbf{S}_r^k}.
\end{aligned} \tag{4.38}$$

Proof. It is easily verified that the twistor operators defined above are lying in the kernel of the Clifford multiplication. It remains to prove the orthogonality

of the projection. But this is clear by the observation that e.g. $p_r^+(\bar{f}_j)$ is the adjoint of $p_{r+1}^-(\bar{f}_j)$ w.r.t. the fibrewise scalar product on spinors. \square

In the next step the absolute value of the twistor operators, applied to a spinor, will be calculated. It is therefore sufficient to consider the following reduced twistor operators:

$$\begin{aligned}\mathcal{D}_X^{\pm+} &= \tilde{\nabla}_{q^-(X)} - \sum_{k=1}^{2m} \frac{1}{A_{r\pm 1, k+1}^{\pm+}} p_{r\pm 1}^{\mp}(q^-(X)) p_r^{\pm}(\bar{f}_k) \tilde{\nabla}_{f_k} \\ \mathcal{D}_X^{\pm-} &= \tilde{\nabla}_{q^+(X)} - \sum_{k=1}^{2m} \frac{1}{A_{r\pm 1, k-1}^{\pm-}} p_{r\pm 1}^{\mp}(q^+(X)) p_r^{\pm}(f_k) \tilde{\nabla}_{\bar{f}_k} .\end{aligned}\quad (4.39)$$

In the rest of this section the constants $A_{r,k}^{\pm\pm}$ are calculated explicitly. In order to do this, some technical lemmata are necessary. In the following calculations dealing with the complex structures J_a , one often has to distinguish the cases $a = 1$ and $a \neq 1$. Therefore the convention will be used that summation over a' means summation over $a = 2$ and $a = 3$ but not over $a = 1$. Moreover, w.r.t. the summation over the indices a and b the Einstein convention is used to avoid too many sums.

Lemma 4.2

$$\sum_j J_{a'} f_j J_{a'} \bar{f}_j = \sum_j \bar{f}_j f_j . \quad (4.40)$$

Proof.

$$\sum_j J_{a'} f_j J_{a'} \bar{f}_j = \frac{1}{4} \sum_j (J_{a'} e_{2j-1} - i J_{a'} e_{2j}) (J_{a'} e_{2j-1} + i J_{a'} e_{2j}) . \quad (4.41)$$

A new base is defined by $e'_j = (-1)^{j+1} J_{a'} e_j$. Hence,

$$J_1 e'_{2j-1} = J_1 J_{a'} e_{2j-1} = -J_{a'} J_1 e_{2j-1} = -J_{a'} e_{2j} = e'_{2j} . \quad (4.42)$$

Therefore

$$\sum_j J_{a'} f_j J_{a'} \bar{f}_j = \frac{1}{4} \sum_j (e'_{2j-1} + i e'_{2j}) (e'_{2j-1} - i e'_{2j}) = \frac{1}{4} \sum_j \bar{f}'_j f'_j . \quad \square \quad (4.43)$$

Lemma 4.3

$$J_{a'} f_j \bar{f}_j = -\bar{f}_j J_{a'} f_j . \quad (4.44)$$

Proof.

$$(J_{a'} e_{2j-1} - i J_{a'} e_{2j}) (e_{2j-1} + i e_{2j}) = -(e_{2j-1} + i e_{2j}) (J_{a'} e_{2j-1} - i J_{a'} e_{2j}) , \quad (4.45)$$

since $J_{a'} e_{2j-1}$ and $J_{a'} e_{2j}$ are orthogonal to e_{2j-1} and e_{2j} . \square

Lemma 4.4

$$\begin{aligned}\sum_j f_j J_{a'} \bar{f}_j &= \frac{1}{2} \tilde{\Omega}_{a'} + \frac{i}{2} \varepsilon_{a'1b'} \tilde{\Omega}_{b'} \\ \sum_j \bar{f}_j J_{a'} f_j &= \frac{1}{2} \tilde{\Omega}_{a'} - \frac{i}{2} \varepsilon_{a'1b'} \tilde{\Omega}_{b'}.\end{aligned}\quad (4.46)$$

Proof.

$$\begin{aligned}\sum_j f_j J_{a'} \bar{f}_j &= \frac{1}{4} \sum_j (e_{2j-1} - ie_{2j}) J_{a'} (e_{2j-1} + ie_{2j}) \\ &= \frac{1}{4} \sum_j (e_{2j-1} J_{a'} e_{2j-1} + ie_{2j} J_{a'} e_{2j-1} \\ &\quad - ie_{2j-1} J_{a'} e_{2j} + e_{2j} J_{a'} e_{2j}) \\ &= \frac{1}{2} \tilde{\Omega}_{a'} + \frac{i}{4} \sum_j (e_{2j-1} J_{a'} e_{2j} - e_{2j} J_{a'} e_{2j-1}) \\ &= \frac{1}{2} \tilde{\Omega}_{a'} + \frac{i}{4} \sum_j (e_{2j-1} J_{a'} J_1 e_{2j-1} + e_{2j} J_{a'} J_1 e_{2j}) \\ &= \frac{1}{2} \tilde{\Omega}_{a'} + \frac{i}{2} \varepsilon_{a'1b'} \tilde{\Omega}_{b'}.\end{aligned}\quad (4.47)$$

The second equation is proven analogously. \square

Lemma 4.5

$$\begin{aligned}\mathcal{J}(f_j) &= \tilde{\Omega}_{a'} J_{a'} f_j + (3 + i\tilde{\Omega}_1) f_j \\ \mathcal{J}(\bar{f}_j) &= \tilde{\Omega}_{a'} J_{a'} \bar{f}_j + (3 - i\tilde{\Omega}_1) \bar{f}_j.\end{aligned}\quad (4.48)$$

Proof.

$$\begin{aligned}\mathcal{J}(f_j) &= \tilde{\Omega}_a J_a f_j + 3f_j = \tilde{\Omega}_{a'} J_{a'} f_j + \tilde{\Omega}_1 J_1 f_j + 3f_j \\ &= \tilde{\Omega}_{a'} J_{a'} f_j + (3 + i\tilde{\Omega}_1) f_j\end{aligned}\quad (4.49)$$

and similarly

$$\begin{aligned}\mathcal{J}(\bar{f}_j) &= \tilde{\Omega}_a J_a \bar{f}_j + 3\bar{f}_j = \tilde{\Omega}_{a'} J_{a'} \bar{f}_j + \tilde{\Omega}_1 J_1 \bar{f}_j + 3\bar{f}_j \\ &= \tilde{\Omega}_{a'} J_{a'} \bar{f}_j + (3 - i\tilde{\Omega}_1) \bar{f}_j.\end{aligned}\quad \square \quad (4.50)$$

In order to make the following expressions more being able to be handled, the notations

$$\begin{aligned}L &= \sum_j \tilde{\Omega}_{a'} f_j J_{a'} \bar{f}_j \\ \bar{L} &= \sum_j \tilde{\Omega}_{a'} \bar{f}_j J_{a'} f_j\end{aligned}\quad (4.51)$$

are introduced.

Lemma 4.6

$$\begin{aligned}
\mathcal{J}(f_j)\bar{f}_j &= -\bar{L} + (3 + i\tilde{\Omega}_1)f_j\bar{f}_j \\
\mathcal{J}(\bar{f}_j)f_j &= -L + (3 - i\tilde{\Omega}_1)\bar{f}_j f_j \\
f_j\mathcal{J}(\bar{f}_j) &= L + (1 - i\tilde{\Omega}_1)f_j\bar{f}_j - 4\bar{f}_j f_j \\
\mathcal{J}(\bar{f}_j)f_j &= \bar{L} + (1 + i\tilde{\Omega}_1)\bar{f}_j f_j - 4f_j\bar{f}_j.
\end{aligned} \tag{4.52}$$

Proof. For example:

$$\begin{aligned}
\mathcal{J}(f_j)\bar{f}_j &= (\tilde{\Omega}_{a'}J_{a'}f_j + (3 + i\tilde{\Omega}_1)f_j)\bar{f}_j \\
&= -\tilde{\Omega}_{a'}\bar{f}_jJ_{a'}f_j + (3 + i\tilde{\Omega}_1)f_j\bar{f}_j \\
&= -\bar{L} + (3 + i\tilde{\Omega}_1)f_j\bar{f}_j
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
f_j\mathcal{J}(\bar{f}_j) &= f_j(\tilde{\Omega}_{a'}J_{a'}\bar{f}_j + (3 - i\tilde{\Omega}_1)\bar{f}_j) \\
&= \tilde{\Omega}_{a'}f_jJ_{a'}\bar{f}_j - 2J_{a'}f_jJ_{a'}\bar{f}_j + (3 - i\tilde{\Omega}_1)f_j\bar{f}_j + 2iJ_1f_j\bar{f}_j \\
&= L + (1 - i\tilde{\Omega}_1)f_j\bar{f}_j - 4\bar{f}_j f_j. \quad \square
\end{aligned} \tag{4.54}$$

Lemma 4.7

$$\begin{aligned}
\sum_j \mathcal{J}(f_j)\mathcal{J}(\bar{f}_j) &= \sum_j \left(-12\bar{f}_j f_j + \tilde{\Omega}_{a'}\tilde{\Omega}_{a'}\bar{f}_j f_j + (-1 + i\tilde{\Omega}_1)L - (1 - i\tilde{\Omega}_1)\bar{L} \right. \\
&\quad \left. + 4L + (3 + i\tilde{\Omega}_1)(1 - i\tilde{\Omega}_1)f_j\bar{f}_j \right).
\end{aligned} \tag{4.55}$$

Proof.

$$\begin{aligned}
\sum_j \mathcal{J}(f_j)\mathcal{J}(\bar{f}_j) &= \sum_j \left(\tilde{\Omega}_{a'}J_{a'}f_j + (3 + i\tilde{\Omega}_1)f_j \right) \left(\tilde{\Omega}_{a'}J_{a'}\bar{f}_j + (3 - i\tilde{\Omega}_1)\bar{f}_j \right) \\
&= \sum_j \left(\tilde{\Omega}_{a'}J_{a'}f_j\tilde{\Omega}_{b'}J_{b'}\bar{f}_j + \tilde{\Omega}_{a'}J_{a'}f_j(3 - i\tilde{\Omega}_1)\bar{f}_j \right. \\
&\quad \left. + (3 + i\tilde{\Omega}_1)f_j\tilde{\Omega}_{a'}J_{a'}\bar{f}_j + (3 + i\tilde{\Omega}_1)f_j(3 - i\tilde{\Omega}_1)\bar{f}_j \right) \\
&= \sum_j \left(\tilde{\Omega}_{a'}\tilde{\Omega}_{b'}J_{a'}f_jJ_{b'}\bar{f}_j - 2\tilde{\Omega}_{a'}J_{b'}J_{a'}f_jJ_{b'}\bar{f}_j \right. \\
&\quad \left. + \tilde{\Omega}_{a'}(3 - i\tilde{\Omega}_1)J_{a'}f_j\bar{f}_j + 2i\tilde{\Omega}_{a'}J_1J_{a'}f_j\bar{f}_j \right. \\
&\quad \left. + (3 + i\tilde{\Omega}_1)\tilde{\Omega}_{a'}f_jJ_{a'}\bar{f}_j - 2(3 + i\tilde{\Omega}_1)J_{a'}f_jJ_{a'}\bar{f}_j \right. \\
&\quad \left. + (3 + i\tilde{\Omega}_1)(1 - i\tilde{\Omega}_1)f_j\bar{f}_j \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_j \left(4i\tilde{\Omega}_1 \bar{f}_j f_j + \tilde{\Omega}_{a'} \tilde{\Omega}_{a'} \bar{f}_j f_j + (3 - i\tilde{\Omega}_1) \tilde{\Omega}_{a'} J_{a'} f_j \bar{f}_j \right. \\
&\quad \left. - 4i\varepsilon_{a'1b'} \tilde{\Omega}_{b'} J_{a'} f_j \bar{f}_j + 2\tilde{\Omega}_{a'} J_{a'} f_j \bar{f}_j \right. \\
&\quad \left. + (3 + i\tilde{\Omega}_1) L - 4(3 + i\tilde{\Omega}_1) \bar{f}_j f_j \right. \\
&\quad \left. + (3 + i\tilde{\Omega}_1)(1 - i\tilde{\Omega}_1) f_j \bar{f}_j \right) \\
&= \sum_j \left(-12\bar{f}_j f_j + (-1 + i\tilde{\Omega}_1) L + \tilde{\Omega}_{a'} \tilde{\Omega}_{a'} \bar{f}_j f_j \right. \\
&\quad \left. - (3 - i\tilde{\Omega}_1) \tilde{\Omega}_{a'} \bar{f}_j J_{a'} f_j + 4i\varepsilon_{a'1b'} \tilde{\Omega}_{b'} J_1 \bar{f}_j J_{a'} J_1 f_j \right. \\
&\quad \left. - 2\bar{L} + (3 + i\tilde{\Omega}_1)(1 - i\tilde{\Omega}_1) f_j \bar{f}_j \right) \\
&= \sum_j \left(-12\bar{f}_j f_j + \tilde{\Omega}_{a'} \tilde{\Omega}_{a'} \bar{f}_j f_j + (-1 + i\tilde{\Omega}_1) L + 4L \right. \\
&\quad \left. - (5 - i\tilde{\Omega}_1) \bar{L} + 4\bar{L} + (3 + i\tilde{\Omega}_1)(1 - i\tilde{\Omega}_1) f_j \bar{f}_j \right) \\
&= \sum_j \left(-12\bar{f}_j f_j + \tilde{\Omega}_{a'} \tilde{\Omega}_{a'} \bar{f}_j f_j + (-1 + i\tilde{\Omega}_1) L - (1 - i\tilde{\Omega}_1) \bar{L} \right. \\
&\quad \left. + 4L + (3 + i\tilde{\Omega}_1)(1 - i\tilde{\Omega}_1) f_j \bar{f}_j \right). \quad \square \quad (4.56)
\end{aligned}$$

Analogously it can be calculated:

Lemma 4.8

$$\begin{aligned}
\sum_j \mathcal{J}(\bar{f}_j) \mathcal{J}(f_j) &= \sum_j \left(-12f_j \bar{f}_j + \tilde{\Omega}_{a'} \tilde{\Omega}_{a'} f_j \bar{f}_j - (1 + i\tilde{\Omega}_1) L + (-1 - i\tilde{\Omega}_1) \bar{L} \right. \\
&\quad \left. + 4\bar{L} + (3 - i\tilde{\Omega}_1)(1 + i\tilde{\Omega}_1) \bar{f}_j f_j \right). \quad \square \quad (4.57)
\end{aligned}$$

Lemma 4.9 *After restriction to \mathbf{S}_r^k one gets:*

$$\begin{aligned}
\tilde{\Omega}_1|_{\mathbf{S}_r^k} &= (2m - 2k) \text{id} \\
\tilde{\Omega}|_{\mathbf{S}_r^k} &= (6m - 4r(r + 2)) \text{id} \\
L|_{\mathbf{S}_r^k} &= (-2r(r + 2) + (m - k)(2m - 2k + 4)) \text{id} \\
\bar{L}|_{\mathbf{S}_r^k} &= (-2r(r + 2) + (m - k)(2m - 2k - 4)) \text{id}. \quad (4.58)
\end{aligned}$$

Proof. Only the last two equations are not obvious.

$$\begin{aligned}
L &= \sum_j \tilde{\Omega}_{a'} f_j J_{a'} \bar{f}_j \\
&= \sum_j \tilde{\Omega}_{a'} \left(\frac{1}{2} \tilde{\Omega}_{a'} + \frac{i}{2} \varepsilon_{a'1b'} \tilde{\Omega}_{b'} \right) \\
&= \frac{1}{2} ((\tilde{\Omega} - 6m) - \tilde{\Omega}_1 \tilde{\Omega}_1) - 2i \tilde{\Omega}_1.
\end{aligned} \tag{4.59}$$

If $\tilde{\Omega}$ and $\tilde{\Omega}_1$ are replaced by the numerical values, the desired result is obtained. The calculation of \bar{L} is almost the same. \square

With these lemmata, one is able to calculate the projector sums

$$\begin{aligned}
\sum_{j=1}^{2m} p_{r+1}^-(f_j) p_r^+(\bar{f}_j) &= \frac{1}{16(r+2)(r+1)} \left((2r+5)(2r+1) f_j \bar{f}_j \right. \\
&\quad \left. + (2r+1) \mathcal{J}(f_j) \bar{f}_j - (2r+5) f_j \mathcal{J}(\bar{f}_j) - \mathcal{J}(f_j) \mathcal{J}(\bar{f}_j) \right)
\end{aligned} \tag{4.60}$$

(and the corresponding expressions for the other three sums) and finally the numerical values of the constants $A_{r,k}^{\pm\pm}$. These are long and ugly calculations so the author has used a computer program.

Proposition 4.1

$$\begin{aligned}
A_{r,k}^{--} &= \sum_{j=1}^{2m} p_{r+1}^-(f_i) p_r^+(\bar{f}_i) |_{\mathbf{S}_r^k} = \frac{(-m+r)(2+k-m+r)}{2(r+1)} \\
A_{r,k}^{+-} &= \sum_{j=1}^{2m} p_{r-1}^+(f_i) p_r^-(\bar{f}_i) |_{\mathbf{S}_r^k} = \frac{(k-m-r)(2+m+r)}{2(r+1)} \\
A_{r,k}^{-+} &= \sum_{j=1}^{2m} p_{r+1}^-(\bar{f}_i) p_r^+(f_i) |_{\mathbf{S}_r^k} = \frac{(-m+r)(2-k+m+r)}{2(r+1)} \\
A_{r,k}^{++} &= \sum_{j=1}^{2m} p_{r-1}^+(\bar{f}_i) p_r^-(f_i) |_{\mathbf{S}_r^k} = \frac{(-k+m-r)(2+m+r)}{2(r+1)}. \quad \square \tag{4.61}
\end{aligned}$$

5 Lower bound of the spectrum of \tilde{D} on P

Let $\psi \in \Gamma(\mathbf{S}(HP))$ be an eigenspinor of \tilde{D} with eigenvalue λ . In $p \in P$ the sum over the squares of the absolute values of $\mathcal{D}_{e_j}^{++} \psi$ is calculated:

$$\sum_{j=1}^{4m} \|\mathcal{D}_{e_j}^{++} \psi\|_p^2 = 2 \sum_{j=1}^{2m} \left\| \left\langle \tilde{\nabla}_{f_j} \psi - \sum_{k=1}^{2m} \frac{1}{A_{r+1,k+1}^{++}} p_{r+1}^{\mp}(f_j) p_r^+(\bar{f}_k) \tilde{\nabla}_{f_k} \psi, \tilde{\nabla}_{f_j} \psi \right\rangle \right\|$$

$$\begin{aligned}
&= 2 \sum_{j=1}^{2m} \|\tilde{\nabla}_{f_j} \psi\|^2 + \frac{1}{2A_{r+1,k+1}^{++}} \|\tilde{D}^{++} \psi\|^2 \\
&\geq 0.
\end{aligned} \tag{5.62}$$

After division by 2 and integration over P the inequality simplifies to

$$0 \leq \sum_j \|\tilde{\nabla}_{f_j} \psi\|^2 + \frac{1}{4A_{r+1,k+1}^{++}} \|\tilde{D}^{++} \psi\|^2. \tag{5.63}$$

Analogous inequalities are obtained by dealing with $\|\mathcal{D}_{e_j}^{+-} \psi\|_p^2$, $\|\mathcal{D}_{e_j}^{-+} \psi\|_p^2$ and $\|\mathcal{D}_{e_j}^{--} \psi\|_p^2$:

$$\begin{aligned}
0 &\leq \sum_j \|\tilde{\nabla}_{\bar{f}_j} \psi\|^2 + \frac{1}{4A_{r+1,k-1}^{+-}} \|\tilde{D}^{+-} \psi\|^2 \\
0 &\leq \sum_j \|\tilde{\nabla}_{f_j} \psi\|^2 + \frac{1}{4A_{r-1,k+1}^{-+}} \|\tilde{D}^{-+} \psi\|^2 \\
0 &\leq \sum_j \|\tilde{\nabla}_{\bar{f}_j} \psi\|^2 + \frac{1}{4A_{r-1,k-1}^{--}} \|\tilde{D}^{--} \psi\|^2.
\end{aligned} \tag{5.64}$$

Now the first and the second resp. the third and the fourth inequality are added and the Weitzenböck formula can be applied:

$$\begin{aligned}
&\sum_j \left(\|\tilde{\nabla}_{f_j} \psi\|^2 + \|\tilde{\nabla}_{\bar{f}_j} \psi\|^2 \right) + \frac{1}{4A_{r+1,k+1}^{++}} \|\tilde{D}^{++} \psi\|^2 + \frac{1}{4A_{r+1,k-1}^{+-}} \|\tilde{D}^{+-} \psi\|^2 \\
&= \frac{1}{2} \left(\|D\psi\|^2 - \frac{\kappa}{4} \|\psi\|^2 \right) + \frac{1}{4A_{r+1,k+1}^{++}} \|\tilde{D}^{++} \psi\|^2 + \frac{1}{4A_{r+1,k-1}^{+-}} \|\tilde{D}^{+-} \psi\|^2 \\
&\geq 0
\end{aligned} \tag{5.65}$$

and in the same manner:

$$\begin{aligned}
&\sum_j \left(\|\tilde{\nabla}_{f_j} \psi\|^2 + \|\tilde{\nabla}_{\bar{f}_j} \psi\|^2 \right) + \frac{1}{4A_{r-1,k+1}^{-+}} \|\tilde{D}^{-+} \psi\|^2 + \frac{1}{4A_{r-1,k-1}^{--}} \|\tilde{D}^{--} \psi\|^2 \\
&= \frac{1}{2} \left(\|D\psi\|^2 - \frac{\kappa}{4} \|\psi\|^2 \right) + \frac{1}{4A_{r-1,k+1}^{-+}} \|\tilde{D}^{-+} \psi\|^2 + \frac{1}{4A_{r-1,k-1}^{--}} \|\tilde{D}^{--} \psi\|^2 \\
&\geq 0.
\end{aligned} \tag{5.66}$$

At this point the problem arises that the expressions $\|\tilde{D}^{\pm\pm} \psi\|^2$ cannot be calculated directly. Nevertheless it is possible to determine them by additional assumptions on the eigenspinor ψ .

To start with an eigenspinor ϕ of \tilde{D}^2 with eigenvalue λ^2 , it can be assumed, that it is localized in a bundle \mathbf{S}_r^k . An eigenspinor of \tilde{D} is determined from this by $\psi := \lambda\phi \pm \tilde{D}\phi$. Clearly, one has $\psi = \psi_{r-1} + \psi_r + \psi_{r+1} \in \mathbf{S}_{r-1} \oplus \mathbf{S}_r \oplus \mathbf{S}_r$ and in addition ψ_{r-1} and ψ_{r+1} are themselves eigenspinors of \tilde{D} . Therefore

$\lambda\psi_{r-1} + \tilde{D}\psi_{r-1} \in \Gamma(\mathbf{S}_{r-1} \oplus \mathbf{S}_r)$ resp. $\lambda\psi_{r+1} + \tilde{D}\psi_{r+1} \in \Gamma(\mathbf{S}_{r+1} \oplus \mathbf{S}_r)$ must be eigenspinors of \tilde{D} with eigenvalue λ . Hence it can be assumed that an eigenspinor of \tilde{D} is localized in two neighbouring subbundles: $\psi \in \mathbf{S}_r \oplus \mathbf{S}_{r+1}$. If one considers the splitting of \mathbf{S} w.r.t. $\tilde{\Omega}_1$ instead of $\tilde{\Omega}$, the same argumentation is valid. This means, it is possible to assume that for an eigenspinor ψ the following holds: $\psi \in \mathbf{S}^k \oplus \mathbf{S}^{k+1}$. Combining both results, one has:

$$\psi \in \Gamma(\mathbf{S}_r^k \oplus \mathbf{S}_{r+1}^k \oplus \mathbf{S}_r^{k+1} \oplus \mathbf{S}_{r+1}^{k+1}). \quad (5.67)$$

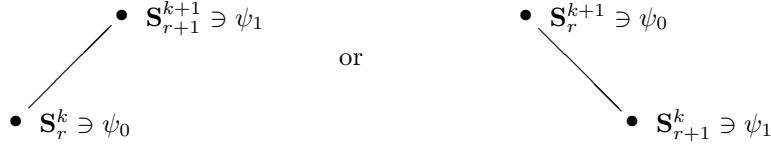
Taking into account the results of Section 3, it can be seen, that in this direct sum only two summands exist, because the eigenvalues for $\tilde{\Omega}_1$ of the partial bundles of \mathbf{S}_r have a distance of 4. Hence, either case A holds:

$$\psi = \psi_0 + \psi_1 \in \Gamma(\mathbf{S}_r^k \oplus \mathbf{S}_{r+1}^{k+1}) \quad \text{if} \quad \frac{k+r-m}{2} \in \mathbb{N}_0 \quad (5.68)$$

or case B:

$$\psi = \psi_0 + \psi_1 \in \Gamma(\mathbf{S}_r^{k+1} \oplus \mathbf{S}_{r+1}^k) \quad \text{if} \quad \frac{k+1+r-m}{2} \in \mathbb{N}_0. \quad (5.69)$$

Clearly, it is assumed that all the bundles \mathbf{S}_r^k appearing in the two cases do exist, i.e. it should not occur that e.g. $\frac{k+1+r-m}{2}$ is an integer but negative. In this case ψ would lie on the allowed lattice but stick out of the allowed region. Now a picture can be helpful again:



Lemma 5.1 *In case A one has:*

$$\begin{aligned} \tilde{D}^{++}\psi_1 &= \tilde{D}^{+-}\psi_1 = \tilde{D}^{-+}\psi_1 = 0, & \tilde{D}^{--}\psi_1 &= \lambda\psi_0 & \text{and} \\ \tilde{D}^{+-}\psi_0 &= \tilde{D}^{-+}\psi_0 = \tilde{D}^{--}\psi_0 = 0, & \tilde{D}^{++}\psi_0 &= \lambda\psi_1. \end{aligned} \quad (5.70)$$

In case B one has:

$$\begin{aligned} \tilde{D}^{++}\psi_1 &= \tilde{D}^{+-}\psi_1 = \tilde{D}^{--}\psi_1 = 0, & \tilde{D}^{-+}\psi_1 &= \lambda\psi_0 & \text{and} \\ \tilde{D}^{+-}\psi_0 &= \tilde{D}^{-+}\psi_0 = \tilde{D}^{--}\psi_0 = 0, & \tilde{D}^{++}\psi_0 &= \lambda\psi_1. \end{aligned} \quad (5.71)$$

Proof. This is trivial, because ψ is an eigenspinor of \tilde{D} , and if one of the mentioned terms would not vanish, it would be orthogonal to ψ .

With this considerations it is now possible to calculate the estimate of the eigenvalue.

Case A: Let $\psi = \psi_0 + \psi_1 \in \Gamma(\mathbf{S}_r^k \oplus \mathbf{S}_{r+1}^{k+1})$ and $\frac{k+r-m}{2} \in \mathbb{N}_0$. Applying inequality (5.65) to ψ_0 leads to (since $\tilde{D}^{+-}\psi_0 = 0$):

$$\lambda^2 \geq \frac{2A_{r+1,k+1}^{++}}{2A_{r+1,k+1}^{++} + 1} \frac{\kappa}{4} = \frac{(-2-k+m-r)(3+m+r)}{(2+r) + (-2-k+m-r)(3+m+r)} \frac{\kappa}{4}. \quad (5.72)$$

In the same manner applying of (5.66) to ψ_1 leads to (since $\tilde{D}^{++}\psi_1 = 0$):

$$\lambda^2 \geq \frac{2A_{r,k}^{--}}{2A_{r,k}^{--} + 1} \frac{\kappa}{4} = \frac{(-m+r)(2+k-m+r)}{r+1+(-m+r)(2+k-m+r)} \frac{\kappa}{4}. \quad (5.73)$$

Both inequalities must hold simultaneously. It is easy to see that in both inequalities the right-hand sides are monotonely decreasing with k . The smallest allowed value of k is $m-r$. For this value the expressions are simplified to:

$$\lambda^2 \geq \frac{2(3+m+r)}{4+2m+r} \frac{\kappa}{4} \quad (5.74)$$

and

$$\lambda^2 \geq \frac{2m-2r}{2m-3r-1} \frac{\kappa}{4}. \quad (5.75)$$

The second inequality is weaker than the first, so it can be omitted. Case B: Let $\psi = \psi_0 + \psi_1 \in \Gamma(\mathbf{S}_r^k \oplus \mathbf{S}_{r+1}^{k-1})$ and $\frac{k+r-m}{2} \in \mathbb{N}_0$. After inserting ψ_0 into (5.65) one gets (now $\tilde{D}^{++}\psi_0 = 0$):

$$\lambda^2 \geq \frac{2A_{r+1,k-1}^{+-}}{2A_{r+1,k-1}^{+-} + 1} \frac{\kappa}{4} = \frac{(-2+k-m-r)(3+m+r)}{(2+r)+(-2+k-m-r)(3+m+r)} \frac{\kappa}{4}. \quad (5.76)$$

And finally one has to apply (5.66) to ψ_1 ($\tilde{D}^{--}\psi_1 = 0$):

$$\lambda^2 \geq \frac{2A_{r,k}^{--}}{2A_{r,k}^{--} + 1} \frac{\kappa}{4} = \frac{(-m+r)(2-k+m+r)}{r+1+(-m+r)(2-k+m+r)} \frac{\kappa}{4}. \quad (5.77)$$

In both inequalities the right-hand sides are monotonely increasing with k . If the maximal allowed $k = m+r$ is considered, the same inequalities as (5.74) and (5.75) are obtained (if it is taken into account that $-\tilde{\Omega}_1$ instead of $\tilde{\Omega}_1$ could have been chosen as distinguished Kähler form, it is clear that case A would have become case B and vice versa, so both cases have to be equivalent).

6 The first eigenvalue of D on M

In the preceeding section an estimate of the first eigenvalue of \tilde{D}^2 on P has been attained. But one is interested in the operator D^2 on M . By Proposition 2.1 it is assured that an eigenvalue of D is an eigenvalue of \tilde{D} too, but surely the converse does not hold. This means that the estimate for \tilde{D} cannot be sharp for D . In addition, the existence of global Kähler structures on P has been used, which do not at all exist on M . The right approach is to look only at horizontal spinors on P that are pull-backs from M , i.e. that are of the form $\pi^*\psi$ with $\psi \in \Gamma(\mathbf{S}(M))$.

If there is given an eigenspinor $\psi \in \Gamma(\mathbf{S}_r(M))$ one has to consider $\tilde{\psi} := \pi^*\psi \in \Gamma(\mathbf{S}_r)$ first and to split $\tilde{\psi}$ into parts which lie in \mathbf{S}_r^k for $(\frac{k+r-m}{2}) \in \mathbb{N}_0$ in order to attain exact estimates. But it will turn out that $\tilde{\psi}$ always has contributions

in the subbundles for maximal resp. minimal k . Hence the estimate calculated in the section above for maximal resp. minimal k does hold.

Now let $\psi \in \Gamma(\mathbf{S}_r)$, $0 \leq r \leq m$, be an eigenspinor of D^2 on M . On P there is a distinguished Kähler form $\tilde{\Omega}_1$ operating by Clifford multiplication on $\tilde{\psi}$. By definition, in a point $p_0 \in P$ it holds:

$$\tilde{\Omega}_1 \tilde{\psi}(p_0) = \frac{1}{2} \sum_{j=1}^{4m} e_j^* J_1 e_j^* \tilde{\psi}(p_0) = \pi^*(\Omega_1(p_0)\psi)(p_0), \quad (6.78)$$

where $\Omega_1(p_0) := \frac{1}{2} \sum_{j=1}^{4m} e_j J_1(p_0) e_j$. Here one can restrict oneself completely to representation theory. For this, one considers the splitting of $(\mathbf{S}_r(M))_{\pi(p_0)}$ w.r.t. Clifford multiplication with $\Omega_1(p_0)$. As seen in Section 3, $(\mathbf{S}_r(M))_{\pi(p_0)} \simeq (\mathbf{S}_r)_{p_0} = \oplus_{s=0}^r (\mathbf{S}_r^{(s)})_{p_0}$ splits into $r+1$ eigenspaces. There is a corresponding splitting of ψ :

$$\psi = \sum_{s=0}^r \psi_{p_0}^{(s)}. \quad (6.79)$$

The index p_0 denotes that the splitting depends on the choice of the three almost-complex structures, i.e. on the point p_0 in the fibre of P .

Under the operation of $\tilde{\Omega}_1$ the eigenspinor $\tilde{\psi}$ of \tilde{D}^2 splits into parts lying in the bundles $\mathbf{S}_r^{(s)}$. The splitting (6.79) corresponds exactly to the splitting of $\tilde{\psi}$ in p_0 . If in (6.79) the summand $\psi_{p_0}^{(0)}$ does not vanish, it is clear that inequality (5.74) for the eigenvalue of $\tilde{\psi}$ must hold, because on P , the subbundles $\mathbf{S}_r^{(s)}$ are invariant under \tilde{D}^2 . The Dirac eigenvalues of $\tilde{\psi}$ and of ψ are the same, so the desired estimate holds if $\psi_{p_0}^{(0)} \neq 0$ has been proven.

Now let $p \neq p_0$ be another point on P in the same fibre as p_0 . Hence, there exist $g \in \mathbf{SO}(3)$ with $p = p_0 g$. There are some consequences for the operation of $\tilde{\Omega}_1$: first, as above

$$\tilde{\Omega}_1 \tilde{\psi}(p_0 g) = \pi^* \left(\frac{1}{2} \sum_{j=1}^{4m} e_j J_1(p_0 g) e_j \psi \right) (p_0 g). \quad (6.80)$$

By definition, $J_1(p_0 g) = g^{-1} J_1(p_0)$ and $\Omega_1(p_0 g) = g^{-1} \Omega_1(p_0)$ hold, where $\mathbf{SO}(3)$ operates on Ω_1 , Ω_2 and Ω_3 in the natural way. As above, there is a splitting of ψ corresponding to the weight space decomposition w.r.t. $g^{-1} \Omega_1(p_0)$:

$$\psi = \sum_{s=0}^r \psi_{p_0 g}^{(s)}. \quad (6.81)$$

In terms of representation theory this means that if $\{H_1, H_2, H_3\}$ forms the standard base of $\mathfrak{so}(3)$ with $[H_a, H_b] = 2\varepsilon_{abc} H_c$, it is clear that Ω_a is the image of $2iH_a$ for $a = 1, 2, 3$ under the representation determined by the choice of p_0 in the given fibre. Complexification of $\mathfrak{so}(3)$ and the definition of $X_1 = \frac{1}{2}(H_2 + iH_3)$ and $Y_1 = \frac{1}{2}(H_2 - iH_3)$ leads to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with the canonical commutator equations. Hence, $\mathbf{S}_r(M)_{\pi(p_0)}$ becomes a representation space of

$\mathfrak{sl}(2, \mathbb{C})$, which will now be identified with an abstract representation space V of highest weight r . This weight is to be regarded as highest eigenvalue of H_1 . If an other point p_0g on the fibre is chosen, r is to be regarded as highest eigenvalue of $g^{-1}H_1$.

For simplicity it can be assumed that V is irreducible. If not, the following is to be carried out for all irreducible parts separately.

Proposition 6.1 *For every $v \in V$ there is a $g \in \mathbf{SO}(3)$ such that v has contributions to the subspace of highest weight w.r.t. gH_1 .*

Proof. Let v be given and $g \in \mathbf{SO}(3)$. Let V be equipped with a norm, and let $\{v_g^s\}$ be a base of normalized vectors spanning the weight spaces V_g^s w.r.t. gH_1 . Then v splits into contributions to weight spaces w.r.t. gH_1 in the following manner:

$$v = \sum_{s=0}^r a_g^s v_g^s, \quad a_g^s \in \mathbb{C}. \quad (6.82)$$

Now let $g_t : [0, 1] \rightarrow \mathbf{SO}(3)$ be a path in $\mathbf{SO}(3)$ and $g_0 = e$. By taking derivatives of (6.82) one obtains:

$$0 = \frac{d}{dt} \Big|_{t=0} v = \sum_{s=0}^r \left(\frac{d}{dt} \Big|_{t=0} (a_{g_t}^s) v_e^s + a_e^s \frac{d}{dt} \Big|_{t=0} v_{g_t}^s \right). \quad (6.83)$$

In order to calculate the derivative of $v_{g_t}^s$ one considers

$$g_t H_1 v_{g_t}^s = (r - 2s) v_{g_t}^s \quad (6.84)$$

and derivatives are taken:

$$\frac{d}{dt} \Big|_{t=0} (g_t H_1) v_e^s + H_1 \frac{d}{dt} \Big|_{t=0} (v_{g_t}^s) = (r - 2s) \frac{d}{dt} \Big|_{t=0} v_{g_t}^s. \quad (6.85)$$

Since $g \in \mathbf{SO}(3)$, $\frac{d}{dt} \Big|_{t=0} (g_t H_1) = A_2 H_2 + A_3 H_3$ for some $A_2, A_3 \in \mathbb{R}$. The path g_t can be chosen such that $A_2, A_3 \neq 0$. On the other side, H_2 and H_3 can be expressed by the ladder operators X_1 and Y_1 w.r.t. H_1 :

$$H_2 = X_1 + Y_1, \quad H_3 = -i(X_1 - Y_1), \quad (6.86)$$

such that after using (6.85) the following holds:

$$((A_2 - iA_3)X_1 + (A_2 + iA_3)Y_1)v_e^s = (r - 2s - H_1) \frac{d}{dt} \Big|_{t=0} v_{g_t}^s. \quad (6.87)$$

The derivative of $v_{g_t}^s$ in $t = 0$ has contributions in V_e^{s-1} and V_e^{s+1} only, explicitly:

$$\begin{aligned} \left[\frac{d}{dt} \Big|_{t=0} v_{g_t}^s \right]_e^{s-1} &= \frac{1}{2} (A_2 - iA_3) X_1 v_e^s \\ \left[\frac{d}{dt} \Big|_{t=0} v_{g_t}^s \right]_e^{s+1} &= -\frac{1}{2} (A_2 + iA_3) Y_1 v_e^s. \end{aligned} \quad (6.88)$$

The proposition follows easily by contradiction. First it is assumed that v does not have contributions to the space of highest weight with respect to gH_1 for all $g \in \mathbf{SO}(3)$. Let $s_0 > 0$ be the largest index satisfying $a_g^s = 0$ for $s < s_0$ and all $g \in \mathbf{SO}(3)$. Without loss of generality $a_e^{s_0} \neq 0$ can be assumed. But looking at the contribution to $V_e^{s_0-1}$ in equation (6.83):

$$0 = \frac{d}{dt} \Big|_{t=0} (a_{g_t}^{s_0-1}) v_e^{s_0-1} + a_e^{s_0} \cdot \frac{1}{2} (A_2 - iA_3) X_1 v_e^{s_0}, \quad (6.89)$$

a contradiction is obtained because by assumption $a_e^{s_0} \neq 0$, but $a_g^{s_0-1} = 0$ for all $g \in \mathbf{SO}(3)$. \square

It is obvious how to proceed further: the eigenspinor $\tilde{\psi} = \pi^* \psi$ has contributions to $\mathbf{S}_r^{(0)} = \mathbf{S}_r^{m-r}$ and hence for λ^2 in the subbundle $\mathbf{S}_r(M)$ it holds:

$$\lambda^2 \geq \frac{2(3+m+r)}{4+2m+r} \frac{\kappa}{4}. \quad (6.90)$$

The right-hand side is monotonly increasing with r , so the universal estimate is obtained by setting $r = 0$:

$$\lambda^2 \geq \frac{m+3}{m+2} \frac{\kappa}{4}. \quad (6.91)$$

7 What comes next?

There is still the open question for which quaternionic Kähler manifolds of positive scalar curvature the obtained lower bound is sharp. Maybe the formulation of a hyperkählerian Killing equation is necessary and the search for criterions of existence of hyperkählerian Killing spinors will give an answer to that question. By analogous methods results are obtained by C. Bär [Bär93] in the case of Riemannian manifolds and by A. Moroianu [Mor94] in the Kählerian case.

References

- [Bär93] C. Bär, Real Killing Spinors and Holonomy, Commun. Math. Phys. 154 (1993) 509–521;
- [Bon67] E. Bonan, Sur les G-structures de type quaternionien, Cahiers Top. et Geom. Diff. 9 (1967), 389–461;
- [Fri80] T. Friedrich, Der erste Eigenwert des Diracoperators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalkrümmung, Math. Nachrichten 97 (1980) 117–146;
- [Hij84] O. Hijazi, Opérateurs de Dirac sur les variétés riemanniennes: Minoration des valeurs propres, Thèse de 3ème Cycle, Ecole Polytechnique (1984);

- [HiM95-1] O. Hijazi, J.-L. Milhorat, Décomposition du fibré des spineurs d'une variété spin Kähler-quaternionienne sous l'action de la 4-forme fondamentale, *Journal of Geometry and Physics* 15 (1995) 320–332;
- [HiM95-2] O. Hijazi, J.-L. Milhorat, Minoration des valeurs propres de l'opérateur de Dirac sur les variétés spin Kähler-quaternioniennes, *J. Math. Pures Appl.* 74 (1995) 387–414;
- [HiM96] O. Hijazi, J.-L. Milhorat, Twistor operators and eigenvalues of the Dirac operator on compact quaternion-Kähler spin manifolds, to appear in *Annals of Global Analysis and Geometry*;
- [Kir86] K. D. Kirchberg, An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature, *Ann. Glob. Analysis and Geometry* Vol. 4, No. 3 (1986) 291–325;
- [Kir90] K. D. Kirchberg, The first eigenvalue of the Dirac operator on Kähler manifolds, *Journ. Geom. Phys* Vol. 7, n. 4 (1990) 447–468;
- [Kra66] V. Y. Kraines, Topology of quaternionic Manifolds, *Trans. American Mathematical Society* 122 (1966) 357–367;
- [Mil92] J.-L. Milhorat, Spectre de l'opérateur de Dirac sur les espaces projectifs quaternioniens, *C. R. Acad. Sci. Paris* 314 (1992) 69–72;
- [Mor94] A. Moroianu, La première valeur propre de l'opérateur de Dirac sur les variétés Kälériennes compactes, *C.R. Acad. Sci. Paris*, t. 319, Série I, p. 1057-1062, 1994;